Assessing views

A key breakthrough in portfolio management theory was the Black-Litterman framework for finding which subjective view of market performance was best supported by empirical data. However, the question remains how to measure the divergence of a single manager view conditioned using this framework with a firm-wide view of the market embodying the equilibrium returns found from data. Here, Gianluca Fusai and Attilio Meucci provide a technique for doing this

The allocation decisions across all the products of an asset management company should reflect the set of views expressed by the asset management team. In practice, this implies a simultaneous optimisation of hundreds of portfolios, each with different characteristics and constraints. Failing to properly structure this process causes inconsistencies and dispersions in the performance across the various lines, with a negative impact on the company's ability to establish its 'brand' in the investment industry.

One way to solve this problem was proposed by Black & Litterman (1992). In their seminal paper, Black & Litterman provided a breakthrough methodology to integrate the committee's views with a firm-wide forecasting model (universal equilibrium in their case). The views are inputted together with a confidence level, which balances the impact of the views on the final asset allocation with respect to the forecasting model: the stronger the confidence, the further the allocation from the optimal allocation based on the forecasting model only. This methodology is very flexible, in that the practitioner does not necessarily need to express views on all the asset classes considered in the forecasting model. Furthermore, the output is model-coherent: optimal asset allocations based on the Black & Litterman approach are not dramatically different from those based on the original firm-wide forecasting model. This is an extremely valuable feature in practical applications.

Practitioners know qualitatively how extreme their views are, that is, how far they are from the forecasting model. Nevertheless, it would be advisable to have at our disposal a single number, an index between zero and one, that quantifies how close the output of the Black & Litterman methodology is to the firm-wide model. Furthermore, we won't know the danger of the view until it is combined with a risk preference on the efficient frontier. However, it would be useful to have indications about which among the views expressed are the furthest from the forecasting model, and how to fix them accordingly. Given the measure in this article, we can tell how extreme the view might be. To the best of our knowledge, no methodology has been developed to answer the above two problems. First, we review the Black & Litterman approach. Then we provide a solution to the above problems, which is based on simple statistical and mathematical techniques. Following that, we see the effects on asset allocation. Then we conclude.

The Black & Litterman approach

In the Black & Litterman framework, the firm-wide accepted forecasting model (or the general equilibrium model in the original article) assumes that the N-dimensional vector of logarithmic returns $\mathbf{r}$ on N asset classes at a given horizon is a multivariate normal distribution:

$$\mathbf{r} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \tag{1}$$

where $\boldsymbol{\mu}$ is the $N \times 1$ vector of expected future returns and $\boldsymbol{\Sigma}$ is the $N \times N$ covariance matrix of future returns. This assumption is consistent with the standard Black-Scholes (1973) model. This means that the density function of the returns is the multivariate Gaussian:

$$G(\mathbf{r};\boldsymbol{\mu},\boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{\frac{d}{2}} |\boldsymbol{\Sigma}|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(\mathbf{r}-\boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{r}-\boldsymbol{\mu})\right\} \tag{2}$$

To illustrate, we consider the oversimplified case of an institution that adopts the RiskMetrics model to optimise the allocation of an international stock fund that invests in the following six stock market national indexes: Italy, Spain, Switzerland, Canada, the US and Germany. In this case, the time horizon of the returns on the above classes is one day and the returns expected value $\boldsymbol{\mu}$ is zero. The covariance matrix of future daily returns on the above classes $\boldsymbol{\Sigma}$ is estimated by exponential smoothing of past daily returns and made publicly available by RiskMetrics. The matrix in our example was estimated in August 1999. Here, we report its decomposition in terms of the vector of volatilities $\boldsymbol{\sigma}$ and the correlation matrix $\mathbf{C}$:

$$\boldsymbol{\sigma} = (1.34\%, 1.52\%, 1.53\%, 1.55\%, 1.82\%, 1.97\%)$$

$$\mathbf{C} = \begin{pmatrix} 1 & 54\% & 62\% & \mathbf{25\%} & 41\% & 59\% \\ & 1 & 69\% & \mathbf{29\%} & 36\% & \mathbf{83\%} \\ & & 1 & \mathbf{15\%} & 46\% & 65\% \\ & & & 1 & \mathbf{47\%} & \mathbf{39\%} \\ & & & & 1 & 38\% \\ & & & & & 1 \end{pmatrix}$$

Typically, a fund manager has personal views on future returns. To continue with our example, he might assess three views: the Spanish index will remain unvaried, the Canadian stock index will score a negative return of 2% and the German index will experience a positive change of 2%. The Spanish and the German markets' stock returns are highly correlated (83%) and the Canadian index is relatively independent of the other markets (bold column/row). A very naive, yet sub-optimal way to include these views in the forecasting model would consist of simply assuming that the returns are still normally distributed:

$$\mathbf{r} \sim N(\boldsymbol{\mu}_{sub}, \boldsymbol{\Sigma}_{sub}) \tag{3}$$

where all the parameters are unchanged:

$$\boldsymbol{\mu}_{sub} \equiv \boldsymbol{\mu}, \qquad \boldsymbol{\Sigma}_{sub} \equiv \boldsymbol{\Sigma} \tag{4}$$

except for substituting in the respective entries of the expected value the personal views:

$$\begin{aligned} [\boldsymbol{\mu}_{sub}]_2 &= 0 \\ [\boldsymbol{\mu}_{sub}]_4 &= -0.02 \\ [\boldsymbol{\mu}_{sub}]_6 &= 0.02 \end{aligned} \tag{5}$$

Unfortunately, the effects on any allocation based on this new model is disruptive: especially in highly correlated markets, extreme corner solutions take place and the overall portfolio becomes meaningless.

The Black & Litterman methodology solves this problem. As a first step, the views are expressed in compact notation as follows:

$$\mathbf{Vr} \sim N(\mathbf{q}, \boldsymbol{\Omega}) \tag{6}$$

where $\mathbf{V}$ is a full rank $K \times N$ matrix such that each of the K rows corresponds to one view and selects the (linear combination of) returns involved in that view; $\mathbf{q}$ is a $K \times 1$ vector that quantifies the views; and $\boldsymbol{\Omega}$ is a $K \times K$ matrix that represents the practitioner's uncertainty about their views. Usually, $\boldsymbol{\Omega}$ is chosen as a multiple of the estimated covariance of the views:

$$\boldsymbol{\Omega} = \alpha \mathbf{V\Sigma V}' \tag{7}$$

where α is a positive scalar.

To continue with our example, the asset manager assesses the views (5) by means of $\mathbf{V}$ and $\mathbf{q}$, which in this case read:

$$\mathbf{V} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{q} = \begin{bmatrix} 0 \\ -2\% \\ 2\% \end{bmatrix} \tag{8}$$

The manager's uncertainty is set comparable to the market volatility, which implies that the uncertainty matrix $\boldsymbol{\Omega}$ is given by (7) with $\alpha = 1$.

Using a Bayesian approach, Black & Litterman prove that the most model-consistent vector of expected returns that incorporates the practitioner's views is a normal distribution:

$$\mathbf{r} \sim N(\mu_{BL}, \Sigma_{BL}) \tag{9}$$

where:

$$\begin{aligned} \mu_{BL} &= \mu + \mathbf{\Sigma V}'(\mathbf{V\Sigma V}' + \boldsymbol{\Omega})^{-1}(\mathbf{q} - \mathbf{V}\mu) \\ \Sigma_{BL} &= \Sigma - \mathbf{\Sigma V}'(\mathbf{V\Sigma V}' + \boldsymbol{\Omega})^{-1}\mathbf{V\Sigma} \end{aligned} \tag{10}$$

Due to the Gaussian setting of the problem in the Black & Litterman approach, the expression of the covariance Σ_{BL} is not affected by the value of the views $\mathbf{q}$, as opposed to the expected values μ_{BL}. Formula (10) is useful in various contexts, among them Markowitz's (1959) optimal portfolio allocation. Inputting expected returns and covariances according to (9) does not give rise to the extreme allocations that one obtains using (3). Finally, as the example shows, the practitioner is not required to express views on all the asset classes.

Assessing the views

The Black & Litterman approach provides the practitioner with a technique to express, in the most model-consistent way possible, their views on the market. Nevertheless, these views might be in strong contrast with the accepted model or, in other words, μ might be very different from μ_{BL}. Is it possible to quantify this difference in statistical terms and provide the asset manager with a simple test, that is, one number that represents the probability of their views if the firm-wide model is true?

To answer this question, we first recall the well-known Z-score, which is widely used by practitioners: the distance of a suspicious value X from the accepted expected value μ divided by the standard deviation σ of X. In our context, the 'suspicious' value is the Black & Litterman vector expected value and the Z-score becomes in this multi-dimensional environment the Mahalanobis distance (Morrison, 1990):

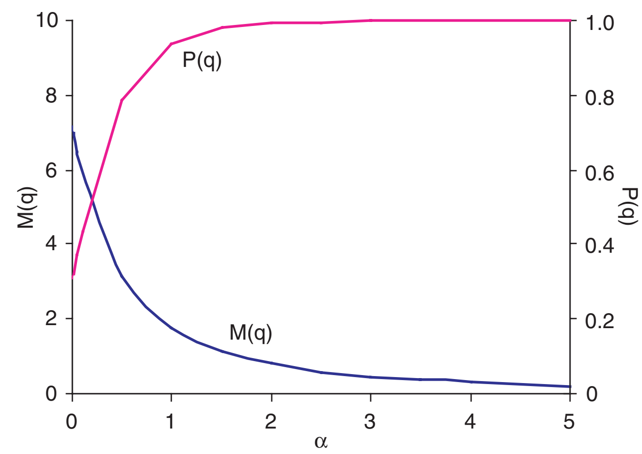
$$M(\mathbf{q}) \equiv (\mu_{BL}(\mathbf{q}) - \mu)' \Sigma^{-1} (\mu_{BL}(\mathbf{q}) - \mu) \tag{11}$$

If the distance $M(\mathbf{q})$ is below a certain threshold, the practitioner can safely place their bets; otherwise, they should be wary that at least one of their views might be too bold. How do we turn the distance $M(\mathbf{q})$ into a probability? We observe that under the hypothesis (1) the Mahalanobis distance (11) is distributed as a chi-square with N degrees of freedom. The number we seek, that is, the probability of the views, is therefore:

$$\mathbb{P}(\mathbf{q}) \equiv 1 - F(M(\mathbf{q})) \tag{12}$$

where F is the cumulative probability of the chi-square distribution with N degrees of freedom.

1. Consistency of the Black-Litterman output with the forecasting model



When the overall probability (12) is below an agreed threshold, often a slight shift in only one of the views is enough to boost this probability upward. Therefore, another natural problem is how to detect the 'boldest' views, and how to fix them accordingly. To solve this problem, we need to calculate the sensitivity of the probability (12) to the views. From the chain rule, the sensitivity is:

$$\begin{aligned} \frac{\partial \mathbb{P}(\mathbf{q})}{\partial \mathbf{q}} &= \frac{\partial \mathbb{P}}{\partial M} \frac{\partial M}{\partial \mu_{BL}} \frac{\partial \mu_{BL}}{\partial \mathbf{q}} \\ &= -2 f(M)(\mathbf{V\Sigma V}' + \boldsymbol{\Omega})^{-1} \mathbf{V}(\mu_{BL} - \mu) \end{aligned} \tag{13}$$

where $f = dF/dM$ is the probability density of the chi-square distribution with N degrees of freedom. To tweak the views, the practitioner simply needs to calculate expression (13) and find the entry with the largest absolute value. If that entry is positive (negative), the respective view must be increased (decreased) slightly.

The probability index (12) is consistent with the role of the manager's uncertainty in (10) and (7) allows us to interpret the uncertainty matrix in (6) and (7). If $\boldsymbol{\Omega}$ is small (α tends to zero in (7)), the practitioner is extremely confident in their views, and indeed the returns satisfy the equation:

$$\mathbf{Vr} \sim N(\mathbf{q}, \mathbf{0}) \tag{14}$$

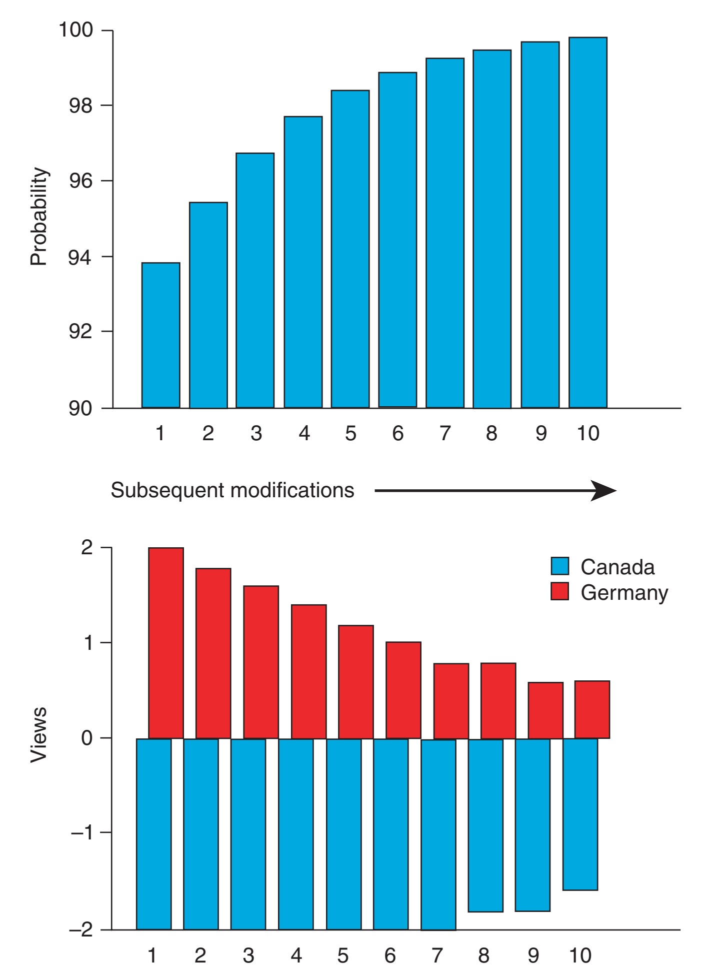
or, in other words, $\mathbf{Vr} = \mathbf{q}$ with certainty. As we can see in figure 1, the output of the Black & Litterman methodology $\mu_{BL}(\mathbf{q})$ is at its furthest from μ, the Mahalanobis distance $M(\mathbf{q})$ (shown on the left-hand scale) approaches its peak and the probability $\mathbb{P}(\mathbf{q})$ (shown on the right-hand scale) approaches its minimum. On the other hand, if $\boldsymbol{\Omega}$ is large (α tends to infinity) the practitioner is completely vague and (9) becomes (1). As a consequence, the distance $M(\mathbf{q})$ tends to zero and the overall probability $\mathbb{P}(\mathbf{q})$ approaches one.

To illustrate how the index works, we apply this recipe to our example. We start with (8). The overall probability (12) and the sensitivities (13) are, respectively:

$$\mathbb{P}(\mathbf{q}) = 93.8\% \qquad \frac{\partial \mathbb{P}(\mathbf{q})}{\partial \mathbf{q}} = \begin{bmatrix} 8.1 \\ 5.6 \\ -9.0 \end{bmatrix} \tag{15}$$

We see that the probability is relatively insensitive to the second view on Canada, even though it is of the same magnitude as the third one on Ger-

2. Progressively tuning the views



many (200 basis points) and that even the first, apparently innocuous, view (0%) on Spain has a higher effect on the probability. This is not unexpected, since the second view refers to a relatively independent market, whereas the first and third view state contrasting opinions on highly correlated markets.

Suppose the manager requires a confidence of at least 95%. To achieve this threshold he should fine-tune, and actually decrease, the third view on the German index. It turns out that a 20bp shift, which changes (8) into:

$$\mathbf{q}_{\text{mod}} = \begin{bmatrix} 0 \\ -2\% \\ 1.8\% \end{bmatrix}$$

brings the overall probability to:

$$\mathbb{P}\left(\mathbf{q}_{\text{mod}}\right) = 95.4\% \tag{16}$$

which is above the desired level.

In figure 2, we see the effect on $\mathbb{P}(\mathbf{q})$ of progressively reducing the boldness of the views. In the lower part, we display different views on the performance of Canada and Germany starting from the initial view +2%, –2%. In the upper part, we report the overall probability corresponding to these less extreme views.

Effects on allocation

The ultimate purpose of the Black & Litterman methodology, and therefore of our technique to improve the consistency of the views, is to obtain optimal portfolios that reflect the management's views in a market-consistent way. To obtain optimal portfolios, we need a definition of optimality that associates the best portfolio with the predictive distribution of returns.

The most standard such rule is the mean-variance approach pioneered by Markowitz (1959). In this approach, the asset management company sells a range of products that span the so-called 'efficient frontier', that is, the set of products with the best trade-off between volatility and expected value of returns. This is the rule that we adopt in this article to illustrate our test. The mean-variance approach is somewhat flawed in this context, in that under any of the assumptions (1), (3) or (9) prices are lognormally distributed, and thus not elliptically, distributed. Alternative ways to obtain optimal portfolios are the minimisation of the value-at-risk, or the minimisation of the expected shortfall.

To find the mean-variance efficient allocations, we need to calculate the expected values and the covariance matrix of the linear returns on assets $\mathbf{R}$, which are linked to the logarithmic returns $\mathbf{r}$ that we have so far dealt with by the relation:

$$\exp(\mathbf{r}) = \mathbf{1} + \mathbf{R}$$

It can be easily proved that, if the logarithmic returns $\mathbf{r}$ are normally distributed:

$$\mathbf{r} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \tag{17}$$

then the linear returns are distributed as a shifted multivariate lognormal, whose expected value and covariance matrix read component-wise respectively:

$$\begin{aligned} \left[E\{\mathbf{R}\}\right]_i &= \exp\left(\mu_i + \frac{1}{2}\boldsymbol{\Sigma}_{ii}\right) - 1 \\ \left[Cov\{\mathbf{R}\}\right]_{ij} &= \exp\left(\mu_i + \frac{1}{2}\boldsymbol{\Sigma}_{ii}\right)\exp\left(\mu_j + \frac{1}{2}\boldsymbol{\Sigma}_{jj}\right)\left(\exp\left(\boldsymbol{\Sigma}_{ij}\right) - 1\right) \end{aligned} \tag{18}$$

At this point for any level of risk aversion $k \in (0, \infty)$, we obtain the respective efficient allocation as the vector of relative weights $\boldsymbol{\omega}(k)$ that solves the following maximisation:

$$\boldsymbol{\omega}(k) \equiv \underset{\boldsymbol{\omega}}{\arg\max}\left\{\boldsymbol{\omega}' E\{\mathbf{R}\} - k\boldsymbol{\omega}' Cov\{\mathbf{R}\}\boldsymbol{\omega}\right\} \tag{19}$$

where the maximisation is subject to the full-investment and the no-short-sale constraints, respectively:

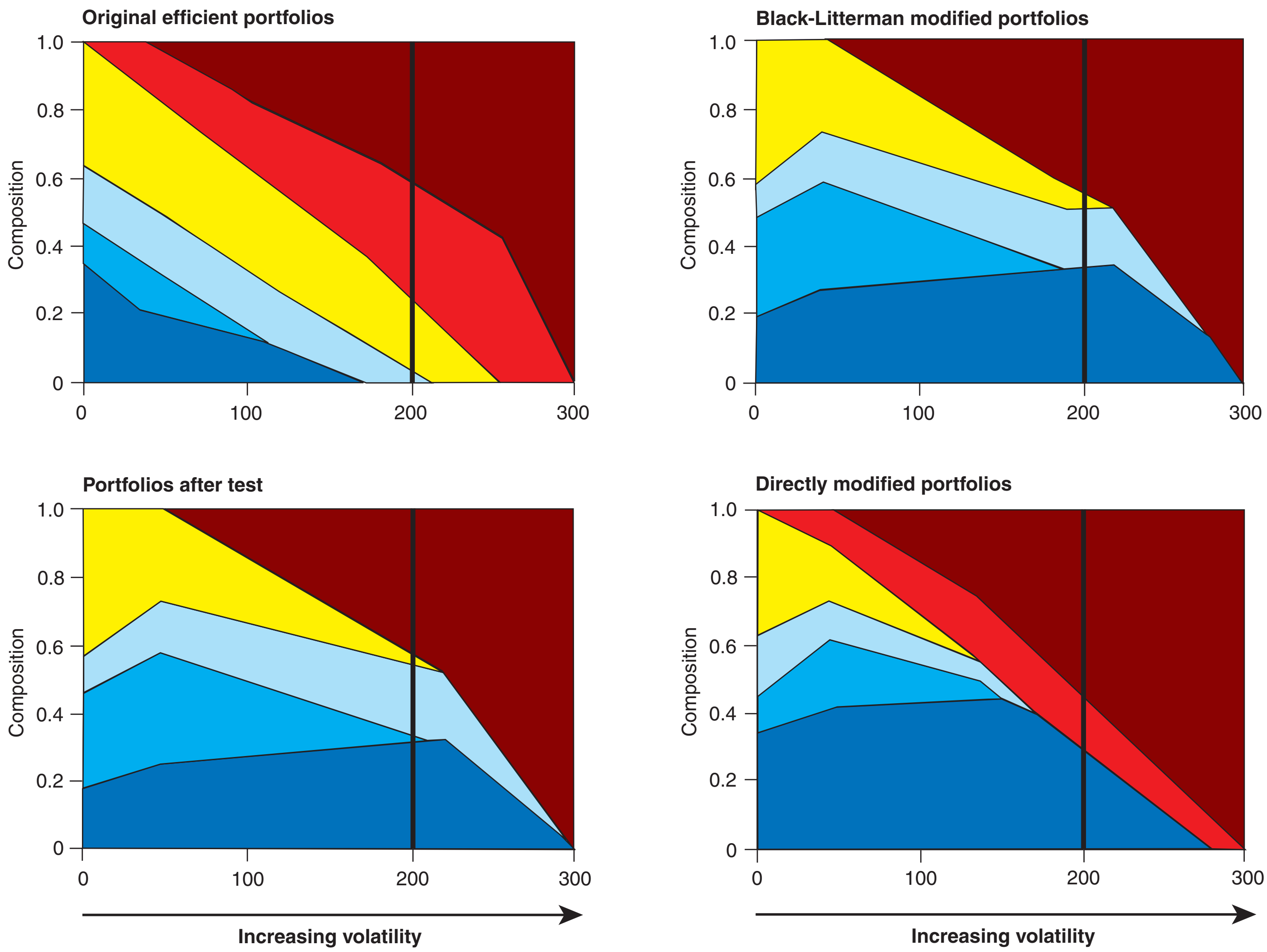
$$\sum_i \omega_i = 1, \qquad \boldsymbol{\omega} \geq \mathbf{0}$$

In figure 3, we plot: the portfolios in the efficient frontier calculated according to the firm-wide forecasting model (1) when no views are inputted; the portfolios in the efficient frontier when the views are inputted by means of the Black & Litterman methodology (9); the portfolios in the efficient frontier when the views are inputted by means of the Black & Litterman methodology after modifying the most extreme view according to our test; and the portfolios in the efficient frontier when the views are inputted by means of the simple substitution (3).

To analyse the result, we focus on the more risky among the efficient allocations, where diversification plays a major role. For example, we consider the allocations corresponding to the vertical bar in the figure. The base-case portfolio, made of four assets, is well diversified. As expected, the Black & Litterman portfolio maintains this degree of diversification, but the allocation promotes the asset relative to the bullish view and demotes the asset relative to the bearish view. The Black & Litterman portfolio modified according to our test makes the portfolio more market-consistent and slightly enhances the diversification effect to five assets. The naive direct modification concentrates the allocation on three assets only.

The test on the views and the recipe to fix them accordingly to obtain

3. Diversification effects of the views on allocation



a more consistent allocation is independent of the final allocation itself. This is an advantage for several reasons. First, the probability index and the test cannot be 'fooled' by the allocation. Suppose that the financial institution invests in two US indexes: the S&P 500 and the Dow Jones. The two assets are different, but very correlated: a portfolio invested, say, 25% in the S&P 500 is basically the same as one that instead invests that portion in the Dow Jones. Even though the weights of the two portfolios display a cumulative difference of 50%, it can be easily tested that the probability of either allocation is substantially the same. Second, the distribution of the Mahalanobis distance of returns is independent of the investor's preferences, whereas the distribution of the portfolio weights, and even more so any test based on this distribution, would depend on the investor utility. Finally, the distribution of the Mahalanobis distance is a simple chi-square, whereas even in the case of a mean-variance manager the distribution of the weights does not admit a simple analytical representation (see, for example, Jobson & Korkie, 1980).

Conclusion

To summarise, the recipe goes as follows: first state the views (6), then calculate the overall probability (12) and the sensitivity (13). If the overall probability is above a certain threshold, go ahead and place your bets. Otherwise, look at the sensitivities. If the largest in absolute value is positive (negative) slightly increase (decrease) the respective view. Repeat the last steps until the overall probability attains the desired threshold. Then proceed to calculate the optimal asset allocation according to a prescribed optimisation rule. ☐

Gianluca Fusai is associate professor of financial mathematics at the University of Piemonte Orientale, Novara, Italy. Attilio Meucci is statistician and trader at Relative Value International, in Greenwich, Connecticut. e-mail: gianluca.fusai@eco.unipmn.it, ameucci@rvifunds.com

References

Black F and R Litterman, 1992
Global portfolio optimization
Financial Analyst Journal, September-October, pages 28–43

Black F and M Scholes, 1973
The pricing of options and corporate liabilities
Journal of Political Economy 81, pages 637–659

Jobson J and B Korkie, 1980
Estimation for Markowitz efficient portfolios
Journal of the American Statistical Association 75, pages 544–554

Markowitz H, 1959
Portfolio selection: efficient diversification of investments
John Wiley & Sons

Morrison D, 1990
Multivariate statistical methods
McGraw-Hill