A Model Calibration

This appendix illustrates the model calibration. Where possible, baseline parameters are chosen in the following such that they correspond to an investment in an average buyout fund.

A. Risk and Return Characteristics

To calibrate the risk and return characteristics of buyout funds, the analysis mainly relies on estimated parameters from Metrick and Yasuda (2010) and Ang et al. (2013). I use a market beta of a private equity fund of $\beta_V = 1.3$ and an alpha of $\alpha_V = 0.04$, which is consistent with recent estimation evidence from Ang et al. (2013), who find average CAPM betas of buyout funds that equal 1.31 and average CAPM alphas that equal 4% per annum.

Metrick and Yasuda (2010) find an annual volatility of $\sigma_i = 60\%$ per individual buyout investment and an average pairwise correlation of $\rho_{ii} = 0.2$ between any two investments. They report that the average buyout fund invests in around 15 companies (with a median of 12). The average holding period of a buyout investment is around four years (see Franzoni et al. (2012)). With a typical fund lifetime of 12 years, the average number of investments running at any time during a fund's lifetime is $N = (15 \times 4)/12 = 5$. Using these values, the total variance of fund returns, σ_V^2, can be approximated by: $\sigma_V^2 = \frac{1}{N}\sigma_i^2(1-\rho_{ii}) + \sigma_i^2\rho_{ii}$, which yields an annual volatility of 0.36, which is rounded to $\sigma_V = 40\%$.

Like Metrick and Yasuda (2010), an annual risk-free rate of $r_f = 5\%$ is used. For the stock market, an annual volatility of $\sigma_M = 15\%$ with an expected return of $\mu_M = 11\%$ is assumed, which implies a risk premium of $\mu_M - r_f = 6\%$. Given this level of stock market volatility, the annual idiosyncratic volatility of a buyout fund is $\sigma_\epsilon = \sqrt{\sigma_V^2 - \beta_V^2\sigma_M^2} = 35\%$, and the correlation between private equity fund returns and aggregate stock market returns yields a reasonable $\rho_V = \beta_V\sigma_M/\sigma_V = 0.49$.

B. Drawdown and Distribution Dynamics

To calibrate the drawdown and distribution rate dynamics, cash flow data at the individual fund level is used, which is net of management fees and carried interest payments. The data has been provided by the Center of Private Equity Research (Cepres), which maintains one of the largest databases of precisely timed deal and fund level cash flows.[1] To minimize a

[1]Earlier versions of this dataset have, for example, been used by Krohmer et al. (2009), Cumming et al.

possible impact by estimated net asset values of unrealized investments, only funds with vintage years from prior to 2001 are being used. This gives a comprehensive sample of monthly cash flows of 200 buyout funds.

To estimate the model parameters, the concept of *conditional least squares* (CLS) is employed. Appendix B derives the least squares estimators for the drawdown rate parameters δ and σ_δ and for the distribution rate parameters ν and σ_ν. Applying these estimators yields the following set of parameters: $\hat{\delta} = 0.41$, $\hat{\sigma}_\delta = 0.21$, $\hat{\nu} = 0.08$, and $\hat{\sigma}_\nu = 0.11$. Simulating the fund cash flows also requires assumptions on the correlations of the drawdown and distribution rate with aggregate market returns. Empirical evidence by Robinson and Sensoy (2011) shows that capital calls and distributions have a systematic component that is pro-cyclical and that distributions are more sensitive to market conditions than calls. In line with these observation, I assume that both correlations are positive and that $\rho_\nu > \rho_\delta$. The specific correlations used in the baseline specification are $\rho_\delta = 0.5$ and $\rho_\nu = 0.8$.

C. Secondary Market Discount Dynamics

To calibrate the secondary market discount rate process, the paper uses the quarterly data on median secondary market discounts for the buyout segment from the Preqin Secondary Market Monitor for the period ranging from March 2003 to March 2013.

Using this data, model parameters κ_π, θ_π, and σ_π are estimated by applying a maximum likelihood (ML) method on a discrete-time counterpart of the Ornstein-Uhlenbeck process (2.3).[2] This approach yields the following parameters for the buyout segment: $\kappa_\pi = 0.42$, $\theta_\pi = 0.16$, and $\sigma_\pi = 0.16$. This set of parameters suggests that the long-run mean of the discount rate equals 16% and that the half-life of the mean-reversion, i.e., the average time it takes the process to get pulled half way back to its long-run mean, equals $\ln 2/\kappa_\pi = 1.65$ years. In addition, the initial discount rate π_0 is set to the long-run mean θ_π. Finally, coefficient ρ_π is estimated by the correlation between quarterly changes in median secondary market discounts and the corresponding quarterly S&P 500 returns. This yields $\rho_\pi = -0.60$, implying that the overall market conditions affect the pricing in the secondary markets and that discounts increase during stock market downturns.

(2010), Franzoni et al. (2012), and Buchner and Stucke (2014).

[2]See Gourieroux and Jasiak (2001), p. 290, for the resulting closed-form estimators.

B Drawdown and Distribution Rate Estimation

This appendix presents the estimation methodology for the parameters of the drawdown and the distribution rate dynamics.

A. Definitions and Methodology

The objective is to estimate model parameters from the observable cash flows of the sample funds at equidistant time points $t_k = k\Delta t$, where $k = 1, \dots, K$, and $K = T/\Delta t$. To make funds of different sizes comparable, the capital drawdowns and capital distributions of all $j = 1, \dots, N$ sample funds are first standardized on the basis of each fund's total invested capital. In the following, let $D_{k,j}$ denote the standardized cumulated capital drawdowns of fund j up to time t_k. Standardized cumulated capital distributions of fund j up to time t_k are denoted by $R_{k,j}$. Finally, $V_{k,j}$ represents the standardized value of fund j at time t_k.

To estimate the model parameters, I use the concept of *conditional least squares* (CLS). The concept of conditional least squares, which is a general approach for estimation of the parameters involved in the conditional mean function $E[X_k|\mathcal{F}_{k-1}]$ of a stochastic process, was given a thorough treatment by Klimko and Nelson (1978). The idea behind the method is to estimate parameters from discrete-time observations $\{X_k\}$ of a stochastic process, such that the sum of squares

$$\sum_{k=1}^{K} (X_k - E[X_k|\mathcal{F}_{k-1}])^2 \tag{B.1}$$

is minimized, where $\mathcal{F}_{k-1}$ is the σ-field generated by $X_1, \dots, X_{k-1}$. This basic idea can be slightly adapted to the particular estimation problem given here. As time-series as well as cross-sectional data of the cash flows of the sample funds is available, a natural idea is to replace X_k in relation (B.1) by the sample average $\bar{X}_k$.

B. Drawdown Rate

To derive an estimator for the drawdown rate δ, let $\bar{D}_k$ denote the sample average of the cumulated capital drawdowns at time t_k, i.e., $\bar{D}_k = \frac{1}{N} \sum_{j=1}^{N} D_{k,j}$. An appropriate goal function to estimate parameter δ is then given by

$$\sum_{k=1}^{K} (\bar{D}_k - E[\bar{D}_k|\mathcal{F}_{k-1}])^2, \tag{B.2}$$

where $\mathcal{F}_{k-1}$ is the σ-field (the available information set) generated by the sequence $\bar{D}_1, \ldots, \bar{D}_{k-1}$. The conditional expectation in (B.2) can, in discrete-time, be states as:

$$E[\bar{D}_k|\mathcal{F}_{k-1}] = \bar{D}_{k-1} + \delta(1 - \bar{D}_{k-1})\Delta t. \tag{B.3}$$

Substituting (B.3) into (B.2), the goal function to be minimized turns out to be

$$\sum_{k=1}^{K} \{\bar{D}_k - \bar{D}_{k-1} - \delta(1 - \bar{D}_{k-1})\Delta t\}^2. \tag{B.4}$$

The least-squares estimator is then the solution to the equation $\sum_{k=1}^{K}(\partial/\partial\delta)\{\bar{D}_k - \bar{D}_{k-1} - \delta(1 - \bar{D}_{k-1})\Delta t\}^2 = 0$. The resulting expression for the estimator is:

$$\hat{\delta} = \frac{1}{\Delta t} \frac{\sum\limits_{k=1}^{K} (\bar{D}_k - \bar{D}_{k-1})(1 - \bar{D}_{k-1})}{\sum\limits_{k=1}^{K} (1 - \bar{D}_{k-1})^2}. \tag{B.5}$$

To estimate the volatility σ_δ, note that the conditional variance of the average capital drawdowns in the interval $(t_{k-1}, t_k]$ can be stated as

$$\begin{aligned} E[\bar{D}_k - E[\bar{D}_k|\mathcal{F}_{k-1}]|\mathcal{F}_{k-1}]^2 &= Var[\bar{\delta}_k(1 - \bar{D}_{k-1})\Delta t|\mathcal{F}_{k-1}] \\ &= [(1 - \bar{D}_{k-1})\Delta t]^2 Var[\bar{\delta}_k|\mathcal{F}_{k-1}]. \end{aligned} \tag{B.6}$$

Using the discrete-time specification of the drawdown rate given in (3.3), the conditional variance $Var[\bar{\delta}_k|\mathcal{F}_{k-1}]$ of the average drawdown rate $\bar{\delta}_k$ is given by:

$$Var[\bar{\delta}_k|\mathcal{F}_{k-1}] = \bar{\sigma}_\delta^2 k \Delta t. \tag{B.7}$$

Substituting Equation (B.3) and (B.7) into (B.6), an appropriate estimator of the variance $\bar{\sigma}_\delta^2$ turns out to be

$$\hat{\bar{\sigma}}_\delta^2 = \frac{1}{K} \sum_{k=1}^{K} \frac{[\bar{D}_k - \bar{D}_{k-1} - \hat{\delta}(1 - \bar{D}_{k-1})\Delta t]^2}{(1 - \bar{D}_{k-1})^2 (\Delta t)^3 k}, \tag{B.8}$$

where $\hat{\delta}$ is evaluated using (B.5). This idea is what Carroll and Ruppert (1988) call the *pseudo-likelihood method*.

Specification (B.8) gives an estimator for the variance of the sample average drawdown rate. Assuming, for simplicity, that the drawdown rates of the sample deals are perfectly positively correlated, it holds that $\hat{\sigma}_{\delta}^2 = \hat{\bar{\sigma}}_{\delta}^2$.

C. Distributions Rate

Following a similar approach to above, an appropriate goal function to estimate the distribution rate ν is given by

$$\sum_{k=1}^{K}(\bar{R}_k - E[\bar{R}_k|\mathcal{F}_{k-1}])^2, \tag{B.9}$$

where $\bar{R}_k$ denotes the sample average of the cumulated capital distributions at time t_k, i.e., $\bar{R}_k = \frac{1}{N}\sum_{j=1}^{N} R_{k,j}$, and $\mathcal{F}_{k-1}$ is the σ-field generated by $\bar{R}_1, \ldots, \bar{R}_{k-1}$.

The conditional expectation in (B.9) can be represented in discrete-time by:

$$E[\bar{R}_k|\mathcal{F}_{k-1}] = \bar{R}_{k-1} + \nu k \bar{V}_{k-1}(\Delta t)^2. \tag{B.10}$$

Substituting Equation (B.10) into (B.9), the goal function to be minimized is given by

$$\sum_{k=1}^{K}\{\bar{R}_k - \bar{R}_{k-1} - \nu k \bar{V}_{k-1}(\Delta t)^2\}^2. \tag{B.11}$$

Similar to above, the least-squares estimator for ν is the solution to the equation $\sum_{k=1}^{K}(\partial/\partial\nu)\{\bar{R}_k - \bar{R}_{k-1} - \nu k \bar{V}_{k-1}(\Delta t)^2\}^2 = 0$. It turns out:

$$\hat{\nu} = \frac{1}{(\Delta t)^2}\frac{\sum\limits_{k=1}^{K}(\bar{R}_k - \bar{R}_{k-1})k\bar{V}_{k-1}}{\sum\limits_{k=1}^{K}k^2\bar{V}_{k-1}^2}. \tag{B.12}$$

The volatility σ_ν can be estimated by first noting that the conditional variance of the average capital distributions in the interval $(t_{k-1}, t_k]$ is given by

$$\begin{aligned} E[\bar{R}_k - E[\bar{R}_k|\mathcal{F}_{k-1}]|\mathcal{F}_{k-1}]^2 &= Var[\bar{\nu}_k \bar{V}_{k-1}\Delta t|\mathcal{F}_{k-1}] \\ &= (\bar{V}_{k-1}\Delta t)^2 Var[\bar{\nu}_k|\mathcal{F}_{k-1}]. \end{aligned} \tag{B.13}$$

Using specification (3.5), the conditional variance $Var[\bar{\nu}_k|\mathcal{F}_{k-1}]$ is given by:

$$Var[\bar{\nu}_k|\mathcal{F}_{k-1}] = \bar{\sigma}_\nu^2 k\Delta t. \tag{B.14}$$

Substituting Equation (B.10) and (B.14) into (B.13), an appropriate estimator of the variance $\bar{\sigma}_\nu^2$ is

$$\hat{\bar{\sigma}}_\nu^2 = \frac{1}{K}\sum_{k=1}^{K} \frac{[\bar{R}_k - \bar{R}_{k-1} - \hat{\nu}k\bar{V}_{k-1}(\Delta t)^2]^2}{(\bar{V}_{k-1})^2(\Delta t)^3 k}, \tag{B.15}$$

where $\hat{\nu}$ is evaluated using (B.12). The average net-asset-values of the funds, $\bar{V}_k$, in (B.12) and (B.15) cannot directly be observed from the underlying cash flow data. I approximate them by the following relationship

$$\bar{V}_k = \sum_{i=1}^{k} (\overline{\Delta D}_i - \overline{\Delta R}_i)(1 + \overline{IRR})^{k-i}, \tag{B.16}$$

where $\overline{IRR}$ is the mean internal rate of return of the sample funds, $\overline{\Delta D}_i = (\bar{R}_i - \bar{R}_{i-1})$ and $\overline{\Delta R}_i = (\bar{R}_i - \bar{R}_{i-1})$.

Similar to above, specification (B.15) only gives an estimator for the variance of the sample average distribution rate. Assuming that the distribution rates of the sample deals are perfectly positively correlated, it also holds here that $\hat{\sigma}_\nu^2 = \hat{\bar{\sigma}}_\nu^2$.

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